Some Issues in Discrete Approximate Solution for Stochastic Differential Equations(Workshop on Stochastic Numerics)

AUTHOR(S):

Komori, Yoshio; Saito, Yoshihiro; Mitsui, Taketomo

CITATION:

Komori, Yoshio ...[et al]. Some Issues in Discrete Approximate Solution for Stochastic Differential Equations(Workshop on Stochastic Numerics). 数理解析研究所講究録 1993, 850: 1-13

ISSUE DATE:

1993-10

URL:

http://hdl.handle.net/2433/83697

数理解析研究所講究録
第850巻 1993年 1-13

Some Issues in Discrete Approximate Solution for Stochastic Differential Equations

小守 良雄 (Yoshio Komori)
名大・工・情報
(Department of Information Engineering
Graduate School of Engineering, Nagoya University)

齊藤 善弘 (Yoshihiro Saito)
聖徳学園女子短大
(Shotoku Gakuen Women's Junior College
1-38 Nakauzura, Gifu-shi)

三井 斌友 (Taketomo Mitsui) *
名大・人間情報学研究科
(Graduate School of Human Informatics,Nagoya University)

Abstract

An evaluation method for numerical schemes of stochastic differential equations is treated. Discussing the source of errors in the discrete numerical solution, we highlight the effect of pseudo-random numbers upon the numerical solution, and point out the significance of the independencies of the series of them required in the numerical schemes. To discriminate the stochastic and deterministic parts in the errors more clearly, we propose a new two-dimensional linear test equation of multiplicative type whose analytical solution can be obtained readily. Our results are illustrated through some numerical examples.

1 Introduction

We are concerned with numerical solutions for initial value problems of stochastic differential equations (SDEs)

$$d\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{y})dt + \boldsymbol{G}(\boldsymbol{y})d\boldsymbol{W}_t, \qquad \boldsymbol{y}_0 = \boldsymbol{c},$$

*Supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japanese Government (No.04302008).

by means of time-discrete approximations. Stochastic analysis gives the precise meaning of the above equation which will be described in the following section, and many physical phenomena are known to be modeled with this equation (*e.g.*[1, 3]). Since the range where we can obtain the analytical solution of SDEs is restricted, we have to apply any numerical approximations to SDEs for the simulation of the phenomena. (For a survey of numerical solutions of SDEs, see [5].) Recent development of computer hardware and software enables us to utilize the computer simulation more as practical means.

However, for the computer simulation of SDEs, the time-discrete approximations as well as the pseudo-random number generators on computer remain some issues to be resolved. In this article we tackle these issues to show some useful results. We are focusing on the numerical schemes of "weak" convergence for SDEs. We will propose a new linear test equation of two-dimenstional multiplicative type. Since the equation has an analytical solution, we can effectively discriminate the discretization error of the scheme from errors caused by other sources. Applying this test equation, we will discuss the affect of the independencies of pseudo-random numbers upon the numerical result of schemes. We will further observe the accuracy bound of numerical result when we use a numerical scheme together with a pseudo-random number generator on a certain computer of finite significant digits.

2 Discrete Approximation for SDEs

For an introduction of basic concepts applied in later sections, we will briefly give some definitions on stochastic differential equations and their discrete approximations.

Let the triplet $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω, $\mathcal{F}$ and P stand for a sample space, a σ-algebra on Ω and a probability measure for $\mathcal{F}$, respectively. In most general form, SDE is defined as follows. For $\omega \in \Omega$, let $\boldsymbol{W}(t,\omega)$ be the m-dimensional Wiener process, and suppose that $\boldsymbol{f}(t,\boldsymbol{y})$ and $\boldsymbol{G}(t,\boldsymbol{y})$ be $\mathcal{F}$-measurable d-dimensional vector-valued and $d \times m$ matrix-valued, respectively, functions defined on the region $[t_0,T] \times \boldsymbol{R}^d$. The initial value problem of *stochastic differential equation of Itô type* is given by

$$\left.\begin{aligned} d\boldsymbol{y}(t,\omega) &= \boldsymbol{f}(t,\boldsymbol{y}(t,\omega))dt + \boldsymbol{G}(t,\boldsymbol{y}(t,\omega))d\boldsymbol{W}(t,\omega), \\ \boldsymbol{y}(t_0,\omega) &= \boldsymbol{c}(\omega), \qquad t_0 \le t \le T < \infty. \end{aligned}\right\} \tag{1}$$

The d-dimensional vector $\boldsymbol{c}(\omega)$ is called the initial data at $t = t_0$ of the equation (1). The above differential equation should be interpreted to the following *stochastic integral equation of Itô type.*

$$\boldsymbol{y}(t,\omega) = \boldsymbol{c}(\omega) + \int_{t_0}^{t} \boldsymbol{f}(t,\boldsymbol{y}(s,\omega))ds \quad + \quad \int_{t_0}^{t} \boldsymbol{G}(s,\boldsymbol{y}(s,\omega))d\boldsymbol{W}(s,\omega), \qquad t_0 \le t \le T < \infty.$$

Here the second integral term in the right-hand side should be taken in Itô's sense.

The d-dimensional stochastic process $y(t,\omega)$ satifying the equation (1) in the sense of stochastic calculus is said to be the solution of the equation. For a sufficient condition for unique existence of the solution, we can mention the following theorem ([1]).

Theorem 1 *Suppose the initial data* $\boldsymbol{c}(\omega)$ *is independent of the stochastic process* $\boldsymbol{W}(t,\omega) - \boldsymbol{W}(t_0,\omega), t \geq t_0$. *If the fucntions* $\boldsymbol{f}, \boldsymbol{G}$ *satisfy the conditions*

$$\|\boldsymbol{f}(t,\boldsymbol{x}) - \boldsymbol{f}(t,\boldsymbol{y})\| + \|\boldsymbol{G}(t,\boldsymbol{x}) - \boldsymbol{G}(t,\boldsymbol{y})\| \leq K\|\boldsymbol{x} - \boldsymbol{y}\|, \quad \forall t \in [t_0,T], \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^d$$

and

$$\|\boldsymbol{f}(t,\boldsymbol{x})\|^2 + \|\boldsymbol{G}(t,\boldsymbol{x})\|^2 \leq K^2(1 + \|\boldsymbol{x}\|^2), \quad \forall t \in [t_0,T], \forall \boldsymbol{x} \in \boldsymbol{R}^d,$$

then the initial value problem (1) has the unique solution $\boldsymbol{y}(t,\omega)$ *on* $[t_0,T]$, *continuous with probability 1.*

Furthermore the stochastic analysis derives the so-called Itô-Taylor expansion for functions of the above solution. This expansion differs much from the conventional deterministic one. Actually the significant point of stochastic analysis lies on this fact.

For the later definition of order of convergence, we assume that along with the Wiener process $\boldsymbol{W}(t,\omega)$ on the probability space there exists a family of non-anticipating σ-subalgebra $\mathcal{F}_t (t_0 \leq t \leq T)$ of $\mathcal{F}$ with respect to $\boldsymbol{W}(t,\omega)$.

As mentioned in the previous section, analytical solutions for SDEs are impossible or hard for many practical problems. Thus we adopt a time-discrete approximation for the solution by considering it upon a partition of the interval $[t_0,T]$ such as

$$t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_K (= T).$$

In the sequel we will take only the equidistant partition, *i.e.* let K be a natural number and set $h = (T - t_0)/K$ and $t_n = t_0 + nh$. First we will introduce the notion of the order of convergence for the discrete approximation.

Let $\boldsymbol{y}(t_n,\omega)$ and $\boldsymbol{y}_n(\omega)$ be the exact and approximate solutions, respectively, for the event ω at time t_n. When the equality $\boldsymbol{y}(t_n,\omega) = \boldsymbol{y}_n(\omega)$ holds, the quantity

$$\boldsymbol{\delta}_{n+1}(\omega) = \boldsymbol{y}(t_{n+1},\omega) - \boldsymbol{y}_{n+1}(\omega) \tag{2}$$

is called the local truncation error of the numerical scheme for ω. Furthermore, if the estimation in terms of conditional expectation

$$\max_{0 \leq n \leq K-1} E[\|\boldsymbol{\delta}_{n+1}(\cdot)\|^2 \mid \mathcal{F}_{t_n}] = O(h^{p+1}), \qquad h \to 0 \tag{3}$$

is achieved, the numerical scheme is called to be of *local order* p in the mean-square sense.

When we do not assume the equality $\boldsymbol{y}(t_n,\omega) = \boldsymbol{y}_n(\omega)$, the difference

$$\boldsymbol{\epsilon}_n(\omega) = \boldsymbol{y}(t_n,\omega) - \boldsymbol{y}_n(\omega), \quad (n = 1,\ldots,K) \tag{4}$$

is called the global error of the numerical scheme for ω. The numerical scheme is said to have p-th *global order of convergence* in the mean-square sense if the estimation

$$\max_{1 \leq n \leq K} E[\|\boldsymbol{\epsilon}_n(\cdot)\|^2 \mid \mathcal{F}_{t_0}] = O(h^p), \qquad h \to 0 \tag{5}$$

is acheived. Note that the consistency notion in the mean-square sense implies the pathwise consistency between the exact and approximate solutions.

The global order of the numerical scheme, which is often referred as the *strong order*, has a very restricted bound. Let F_K be the σ-algebra generated by the discretized m-dimensional Wiener process $\boldsymbol{W}(t_i,\omega)(i=0,\ldots,K-1)$. Among the F_K-measurable approximate solutions for the exact solution $\boldsymbol{y}(t,\omega)$, it is known that the best approximate solution is $E[\boldsymbol{y}(t,\omega) \mid F_K]$. Then for the initial value problem (1), suppose that the functions $\boldsymbol{f}$ and $\boldsymbol{G}$ satisfy the conditions in Theorem 1. Let us refer the following identity as the *Condition A*.

$$\sum_{l=1}^{d} \frac{\partial \boldsymbol{G}_{ij}}{\partial \boldsymbol{y}_l} \boldsymbol{G}_{lk} = \sum_{l=1}^{d} \frac{\partial \boldsymbol{G}_{ik}}{\partial \boldsymbol{y}_l} \boldsymbol{G}_{lj}, \qquad \forall i,j,k$$

We have the following theorem([3, 4]).

Theorem 2 *The approximate solution $E[\boldsymbol{y}(t,\omega) \mid F_K]$ can attain the strong order 2 if the Condition A holds, otherwise it only attains the strong order 1.*

Although the strong order of the numerical scheme is really a desired property, we have to look for a more relaxed concept of convergence to overpass the order barrier mentioned above. This is the *weak order* of convergence. Let $C_P^l(\boldsymbol{R}^d, \boldsymbol{R})$ be the totality of l times continuously differentialble functions, all of whose partial derivatives of order less than or equal to l have polynomial growth. The numerical scheme is said to be of weak order β if the estimation

$$\max_{0\le n<K-1} \|E[\boldsymbol{g}(\boldsymbol{y}(t_n,\cdot) \mid \mathcal{F}_{t_0}] - E[\boldsymbol{g}(\boldsymbol{y}_n(\cdot)) \mid \mathcal{F}_{t_0}]\| = O(h^\beta), \qquad h \to 0 \tag{6}$$

holds for a natural number β and any $\boldsymbol{g} \in C_P^{2(\beta+1)}$.

3 Numerical Behaviour of Runge-Kutta Schemes of High Weak Order

There are several numerical schemes which are designed to have a higher weak order. In the sequel we will restrict ourselves in the case of the scalar Wiener process. Namely, we will consider the initial value problem of SDEs of Itô type

$$\left.\begin{aligned} d\boldsymbol{y}(t,\omega) &= \boldsymbol{f}(\boldsymbol{y}(t,\omega))dt + \boldsymbol{G}(\boldsymbol{y}(t,\omega))dW(t,\omega), \\ \boldsymbol{y}(t_0,\omega) &= \boldsymbol{c}(\omega), \qquad t_0 \le t \le T < \infty. \end{aligned}\right\} \tag{7}$$

in place of (1). PLATEN (p.485 in [5]) propsed the following explicit Runge-Kutta scheme for (7).

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \frac{1}{2}(\boldsymbol{f}(\hat{\boldsymbol{y}}_n) + \boldsymbol{f}(\boldsymbol{y}_n))h \\ &\quad + \frac{1}{4}(\boldsymbol{G}(\boldsymbol{y}_n^+) + \boldsymbol{G}(\boldsymbol{y}_n^-) + 2\boldsymbol{G}(\boldsymbol{y}_n))\Delta W \\ &\quad + \frac{1}{4}(\boldsymbol{G}(\boldsymbol{y}_n^+) - \boldsymbol{G}(\boldsymbol{y}_n^-))\{(\Delta W)^2 - h\}h^{-\frac{1}{2}}, \\ \hat{\boldsymbol{y}}_n &= \boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)h + \boldsymbol{G}(\boldsymbol{y}_n)\Delta W, \\ \boldsymbol{y}^\pm &= \boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)h \pm \boldsymbol{G}(\boldsymbol{y}_n)\sqrt{h}. \end{aligned} \tag{8}$$

Here, h is the stepsize and ΔW, which stands for the difference $W(t_{n+1},\omega) - W(t_n,\omega)$, is realized by the pseudo-random number whose expectaion and covarience are 0 and h, respectively. This scheme is proved to be of weak order 2.

Using this scheme, we carried out a numerical test on the scalar equation of Itô type

$$dy(t,\omega) = ay(t,\omega) + by(t,\omega)dW(t,\omega) \quad (0 < t < T), \quad y(0) = y_0. \tag{9}$$

Since a, b and y_0 are assumed to be constants, the expectation of the solution of (9) is easily given as

$$E[y(t,\cdot)] = y_0 e^{at}.$$

Let $y_h(T)$ be a sample value of numerical solution by a certain numerical scheme with stepsize h at T, then the difference

$$\mu(h) = \langle y_h(T) \rangle - E[y(T,\cdot)]$$

turns out to the mean error at T. Here and in the sequel, the notion $\langle\cdot\rangle$ stands for the arithmetic mean of samples generated by the pseudo-random numbers realizing ΔW.

For the case of $y_0 = 0.1$, $a = 1.5$, $b = 0.01$ and $T = 1.0$, we carried out the numerical simulation to get the 90% confidence interval for the mean error $\mu(h)$ by 20 batches each with 100 trajectories, varying the stepsize $h = 1/2, 1/2^2, \ldots$ and so on. Fig. 1 shows the confidence interval (left) and its midpoint (right) for each h.

Next, for comparison's sake, we carried out numerical computations with an analogue of the classical Runge-Kutta method for ordinary differential equations simply appended with the stochastic term. That is,

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \frac{1}{6}h(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4) + \boldsymbol{G}(\boldsymbol{y}_n)\Delta W, \\ \boldsymbol{k}_1 &= \boldsymbol{f}(\boldsymbol{y}_n), \quad \boldsymbol{k}_2 = \boldsymbol{f}(\boldsymbol{y}_n + \frac{1}{2}h\boldsymbol{k}_1), \\ \boldsymbol{k}_3 &= \boldsymbol{f}(\boldsymbol{y}_n + \frac{1}{2}h\boldsymbol{k}_2), \quad \boldsymbol{k}_4 = \boldsymbol{f}(\boldsymbol{y}_n + h\boldsymbol{k}_3) \end{aligned} \tag{10}$$

for (7). As a matter of course, the weak order of this scheme is 1.

The numerical result when the scheme (10) was applied to the problem (9) with the same data as in the previous case is given in Fig. 2. As in Fig. 1, the Figure shows the tendency of the 90% confidence interval (left) and its midpoint (right) of the mean error at T versus the variation of h. We can observe that while the stepsize h is comparatively large, the mean error looks to behave as if $O(h^2)$. Restricting ourselves on this example, we could not discriminate the difference of weak order between schemes (8) and (10). However we were afraid that this phenomenon comes from the smallness of b. (Remark $b = 0.01$.) So, putting $b = 1.0$, we executed the computation again for (9) with the Runge-Kutta scheme (8). The result is shown in Fig.3, which no longer confirms that the scheme is of weak order 2. The above computations suggest that a mindless test leads to an incorrect conclusion even for numerical schemes of higher weak order. We will seek out the reason and raise a new test equation in the following sections.

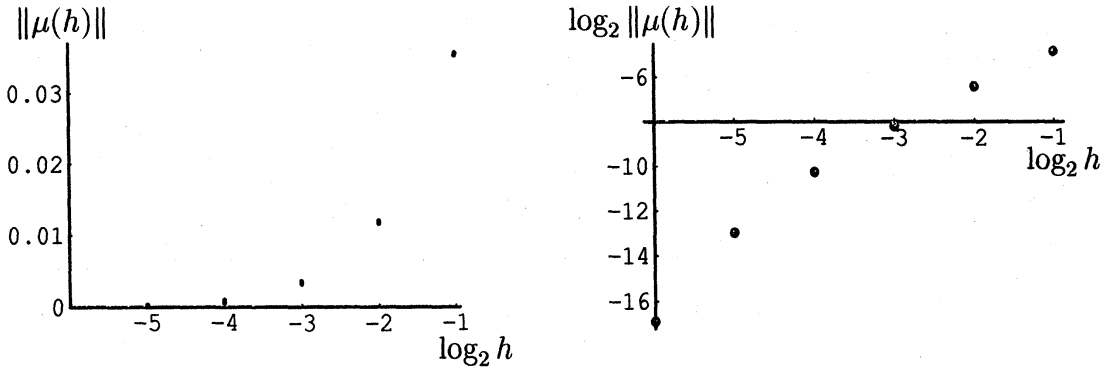

Figure 1: The tendency of the mean error by the Runge-Kutta scheme of weak order 2 ($b = 0.01$)

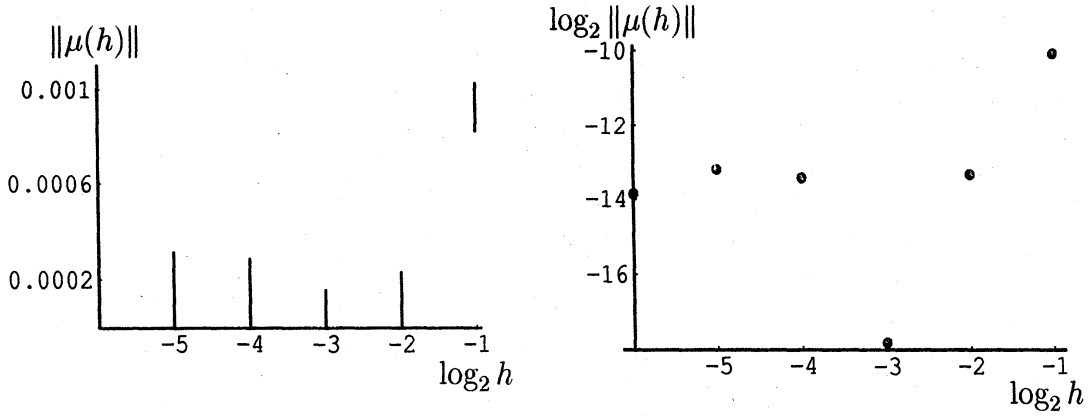

Figure 2: The tendency of the mean error by the classical Runge-Kutta scheme ($b = 0.01$)

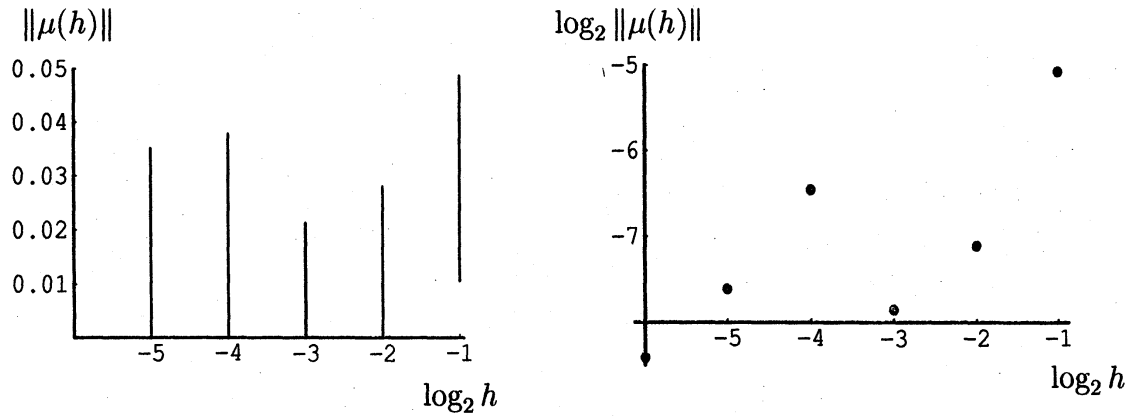

Figure 3: The tendency of the mean error by the weak Runge-Kutta scheme ($b = 1$)

4 Multiplicative Linear Test Equation

The test equation (9) is actually often used to examine the performance of various numerical schemes. Furthermore some numerical stability concepts base on this test equation. However it is obviously insufficient to know the property of shcemes when they are applied to vector-valued SDEs.

In linear vector-valued SDEs, the expectation or the covariance of the solution are easily known to obey ordinary differential equations (ODEs). The ODEs, which can be solved analytically or numerically if necessary, give the time-varying information of the expectation or the covariance, but it is for the whole inspection objects, namely for the *population*. On the other hand, from the numerical solutions of SDEs we can obtain only numerical evaluations for the expectation or the covariance for a certain set of samples out of the population. Therefore, the error we have in hand includes the statistical part together with its counterpart which is intrinsic in the numerical scheme. To estimate the intrinsic error of the numerical scheme, we have to discriminate it from the statistical part of the error. SDEs, whose solution gives its expectation or covariance on a ceratain sample set correctly, should be therefore the one to be solved explicitly. Then we can get the expectation or the covariance upon the set of samples, and compare it with one of numerical solution. We can hardly, however, find an example meeting this purpose in the literatures.

We are here giving a 2-dimensional stochastic differencial equation which can serve the need. Consider the SDE

$$d\boldsymbol{y}(t,\omega) = \boldsymbol{A}\boldsymbol{y}(t,\omega)dt + \boldsymbol{B}\boldsymbol{y}(t,\omega)dW(t,\omega), \tag{11}$$

where the constant matrices A and B are given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ \beta & \gamma \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

Notice that the Wiener process is taken as scalar.

The condition that the both eigenvalues λ of the matrix $\boldsymbol{A}$ should be $\mathrm{Re}\lambda < 0$ imposes the restrictions $\beta < 0$ and $\gamma < 0$, which are assumed hereafter. Denote $\Delta = t - t_0$, then the solution of (11) is expressed as

$$\boldsymbol{y}(t,\omega) = -\frac{1}{4S_q}\begin{bmatrix} 2\gamma\Lambda^- - 2S_q\Lambda^+ & -4\Lambda^- \\ -4\beta\Lambda^- & -2\gamma\Lambda^- - 2S_q\Lambda^+ \end{bmatrix}\boldsymbol{y}(t_0,\omega), \tag{12}$$

where

$$p = -\frac{\alpha^2}{2}\Delta + \alpha\Delta W, \qquad \Delta W = W(t,\omega) - W(t_0,\omega),$$

$$\lambda_1 = p + \frac{\gamma\Delta + S_q\Delta}{2}, \qquad \lambda_2 = p + \frac{\gamma\Delta - S_q\Delta}{2},$$

$$\Lambda^+ = e^{\lambda_1\Delta} + e^{\lambda_2\Delta}, \qquad \Lambda^- = e^{\lambda_1\Delta} - e^{\lambda_2\Delta}, \qquad S_q = \sqrt{\gamma^2 + 4\beta}.$$

First we numerically solve the equation (11) by the Runge-Kutta scheme (8) of weak order 2 and evaluate the arithmetic mean of the numerical solution. Then we calculate the arithmetic

mean of samples from the analytical solution (12) by substituting just the same discretized Wiener process as in the numerical solution to (12). Let $\hat{\boldsymbol{y}}_h(t)$ be a sample value when the discretized Wiener process of stepsize h is substituted into the analytical solution (12) at time t instead of ΔW, and define

$$\nu(h) = \log_2 \|\langle \hat{\boldsymbol{y}}_h(T) \rangle - \langle \boldsymbol{y}_h(T) \rangle\|. \tag{13}$$

In Fig.4 we show the tendency of $\nu(h)$ versus h in the case of $\alpha = 1, \beta = -\frac{1}{4}, \gamma = -3$ and $T = 1$, averaged over 20000 trajectories. This result still does not suggest that the Runge-Kutta scheme (8) has the weak order 2.

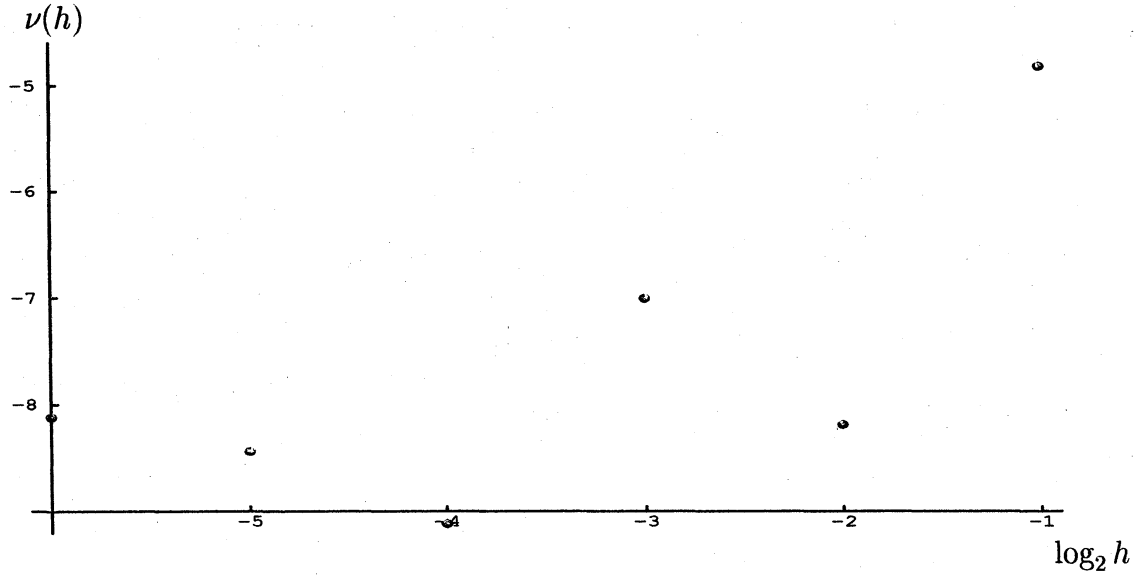

Figure 4: The tendency of $\nu(h)$ versus $\log_2 h$ in the Runge-Kutta scheme

5 Independency of Pseudo-Random Numbers

One possible reason why the numerical behaviour of solution of the Runge-Kutta scheme (8) did not exhibit the weak order 2 is due to the issue of pseudo-random numbers. As a matter of fact, the numerical schemes for SDEs always require the *random numbers* which corresponds to the realization of the Wiener process. However, a complete set of random numbers cannot be generated with finite calculations by computer. Thus we cannot help but use *pseudo-random numbers* in place of random numbers. The pseudo-random numbers are believed to be mutually independent. But the independent pseudo-random numbers themselves are very hard to be generated. Thus the defective set of random numbers adpoted in practical calculations has to be suspected. We will discuss this issue.

The independency of random numbers X and Y which have the normal distribution is equivalent to the uncorrelatedness, *i.e.*

$$E[XY] = E[X]E[Y].$$

Therefore, the test of independency for normal pseudo-random numbers ξ_n and ξ_{n+1} can be done by checking the inequality

$$|\langle \xi_n \xi_{n+1} \rangle - \langle \xi_n \rangle \langle \xi_{n+1} \rangle| < \varepsilon \tag{14}$$

for a prescribed tolerance level ε. Fig. 5 shows the level of the defect of independency of the pseudo-random numbers used in Fig.4. That is, it indicates the quantity

$$MaxEp = \max_n |\langle \xi_n \xi_{n+1} \rangle - \langle \xi_n \rangle \langle \xi_{n+1} \rangle|.$$

versus $\log_2 h$. The Figure tells us that the smaller the stepsize becomes, the more defective the pseudo-random numbers turn out to with respect to their independency.

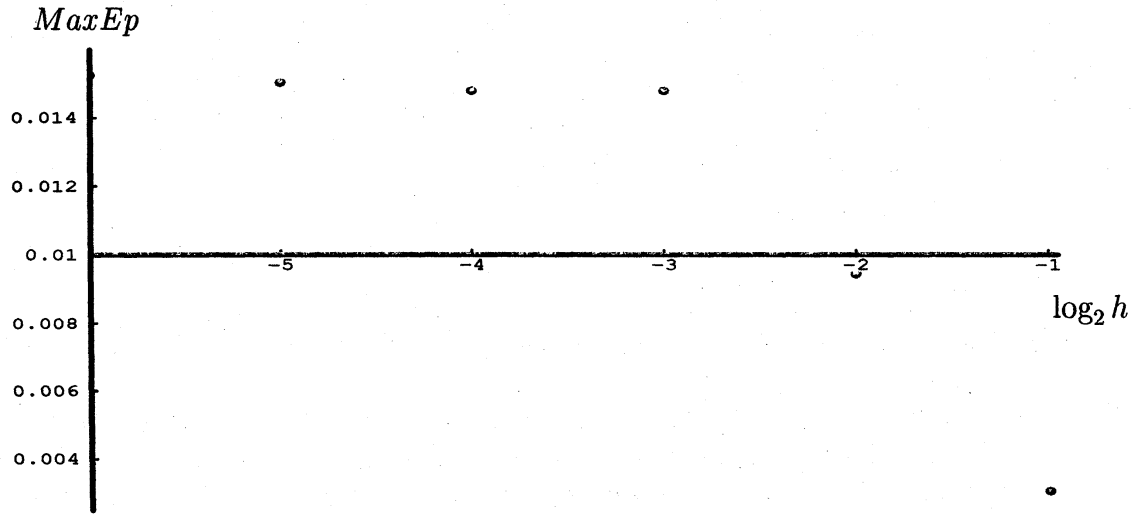

Figure 5: The level of defects of pseudo-random numbers

The above test suggests that we have to select the pseudo-random numbers generated by a naïve method so as to keep the level of defect of their independency within a certain tolerance. Appending the "sieving" process to the pseudo-random number generator under the tolerance level $\varepsilon = 0.004$ in (14), we carried out again the numerical solution of the test equation (11) by the Runge-Kutta method (8) and plotted the tendency of $\nu(h)$ versus $\log_2 h$ in Fig.6. This is the case of $\alpha = 1, \beta = -\frac{1}{4}, \gamma = -3$ and 20000 trajectories. It confirms that the error of the scheme actually behaves as of weak order 2.

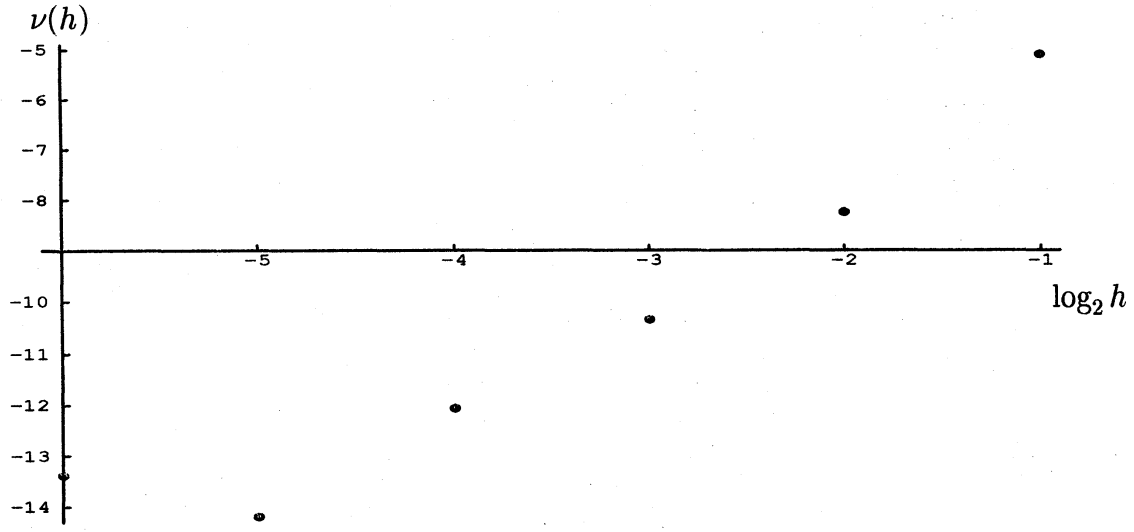

Figure 6: The tendency of $\nu(h)$ versus $\log_2 h$ in the Runge-Kutta scheme with the selected pseudo-random numbers

6 Error Level of Computer-Simulated Solutions

In the previous section, we showed that keeping the independency of pseudo-random numbers within a certain tolerance level is a significant factor to realize the expected weak convergence of numerical solutions supplied by computer simulation. Here we will investigate more the relationship between errors and pseudo-random numbers through numerical experiments for the 2-dimensional test equation.

The expectation of the solution (12), denoted by $\boldsymbol{M}(t) = E[\boldsymbol{y}(t,\cdot)]$, obeys the following ODEs derived from the test equation (11).

$$d\boldsymbol{M}(t) = \boldsymbol{A}\boldsymbol{M}(t)dt. \tag{15}$$

The solution of this equation is given by

$$\boldsymbol{M}(t) = -\frac{1}{4S_q}\begin{bmatrix} 2\gamma\tilde{\Lambda}^- - 2S_q\tilde{\Lambda}^+ & -4\tilde{\Lambda}^- \\ -4\beta\tilde{\Lambda}^- & -2\gamma\tilde{\Lambda}^- - 2S_q\tilde{\Lambda}^+ \end{bmatrix}\boldsymbol{M}(t_0), \tag{16}$$

where the constants are given as follows.

$$\mu_1 = \frac{\gamma + S_q}{2}, \qquad \mu_2 = \frac{\gamma - S_q}{2},$$

$$\tilde{\Lambda}^+ = e^{\mu_1\Delta} + e^{\mu_2\Delta}, \qquad \tilde{\Lambda}^- = e^{\mu_1\Delta} - e^{\mu_2\Delta}, \qquad S_q = \sqrt{\gamma^2 + 4\beta}.$$

Remark that in the solution formula (12) for any $t > t_0$ $\boldsymbol{y}(t,\omega)$ depends only on the difference of the values of the Wiener process at t and t_0. This implies that the solution as well as its expectation can be obtained for the test equation (11) neglecting the intermediate Wiener process between t_0 and t. Thus we can evaluate three kinds of expectations for the test equation.

1. The *analytical* expectation $\boldsymbol{M}(t)$ from (16). We call it the *expectation of the population.*

2. The arithmetic mean of $\hat{\boldsymbol{y}}_h(t)$. This is called the *exact mean with respect to the sample.*

3. The arithmetic mean of the numerical solution of the equation (11) with the scheme (8). This is called the *mean of the numerical solution.*

We should use the same discretized Wiener process in the calculations for the means of the sample and of the numerical solution. To discriminate various factors in the errors, we evaluate the following quantities.

$$\begin{aligned} \boldsymbol{\mu}(h) &= \langle \boldsymbol{y}_h(T) \rangle - \boldsymbol{M}(T), \\ \nu(h) &= \log_2 \| \langle \hat{\boldsymbol{y}}_h(T) \rangle - \langle \boldsymbol{y}_h(T) \rangle \|, \\ \rho(h) &= \log_2 \left(\frac{\| \boldsymbol{M}(T) - \langle \boldsymbol{y}_h(T) \rangle \|}{\| \boldsymbol{M}(T) \|} \right). \end{aligned}$$

The quantity $\boldsymbol{\mu}(h)$ is what we most want to know, but, as discussed in the previous sections, it includes the stochastic and the deterministic parts ([6]) in a non-separating manner. The quantity $\nu(h)$ can be estimated when we know the weak order of the numerical scheme and we take the pseudo-random numbers selected to keep their mutual independency. The third quantity $\rho(h)$ corresponds to the stochastic part, that is the relative eror by using the pseudo-random numbers in place of the complete random numbers. In Fig. 7 we plot $\rho(1/8)$ versus the number of samples N_r applied in the numerical simulations, without the independency check of pseudo-random numbers. The constants adopted in the test equation are $\alpha = 1, \beta = -\frac{1}{4}, \gamma = -3$. The result shows that $\rho(h)$ is certainly affected by the statistical property of pseudo-random numbers. We can observe that the level of $\rho(h)$ tends to be smaller as the number of trajectories increases, but it cannot fall below a ceratin positive level. Thus we are to be ready to allow the contamination of the error caused by $\rho(h)$ into $\boldsymbol{\mu}(h)$.

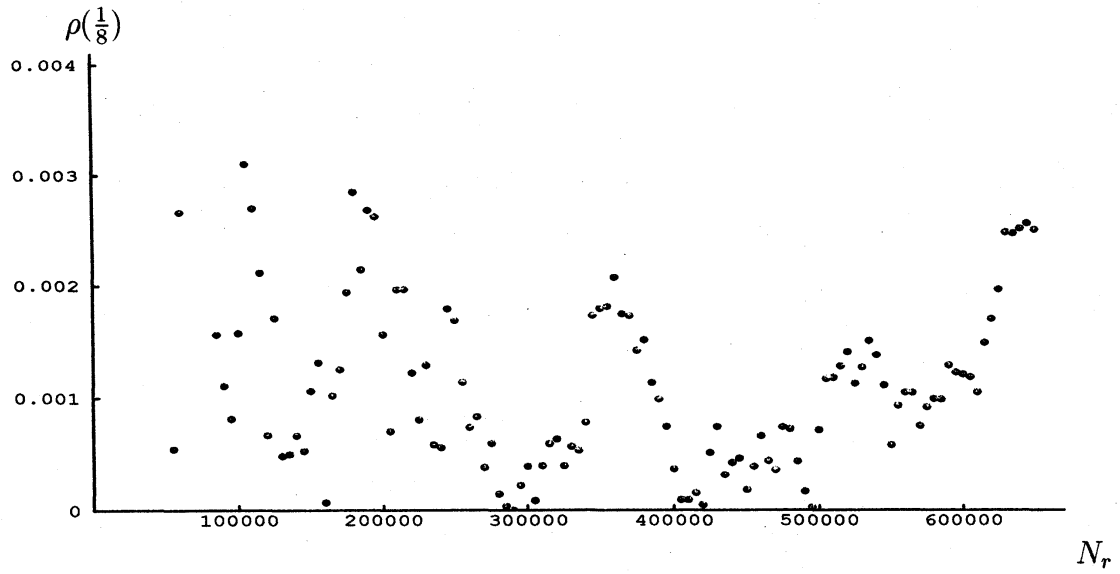

Figure 7: The relative error level of the numerical solutions

7 Concluding Remarks

In the present paper we brought up some issues on the error of numerical solutions for SDEs. First, we have to be careful on the pseudo-random number generators needed in the time stepping process. The weak convergence rate of numerical schemes would not appear correctly for a broader class of SDEs, if we do not use pseudo-random numbers so as to keep their mutual independency within an appropriate level. The "sieving" process should be carried out for them at every time step to realize the difference of the Wiener process. It is however much time-consuming.

Second, for the purpose to discriminate various factors in the global error of the numerical solution, we gave a new two-dimensional multiplicative linear test equation of Itô-type. Since it has the analytical solution, we are able to obtain the arithmetic mean of the realized exact solution along with any discretization of the Wiener process, together with the exact solution of ODEs which the *exact* expectation of analytical solution obeys. Although we do not mention here, similar arguments can be derived for the covariance of solution. The test equation, therefore, useful for the analysis of weak convergence of numerical schemes.

Applying above ways of resolution, we are studying a Rosenbrock-type numerical schemes ([2]) of high weak order. Our result will be forthcoming. On the other hand, we believe the test equation will give a new insight for the stability analysis of numerical schemes, which two of the present authors have developed ([7, 8]).

References

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications,* John Wiley & Sons, New York, 1974.

[2] S.S. Artemiev and I.O. Shukurko, Numerical analysis of dynamics of oscillatory stochastic systems, *Soviet J. Numer. Anal. Math. Modelling,* **6,** 277-298 (1991).

[3] T.C. Gard, *Introduction to Stochastic Differential Equations,* Marcel Dekker, New York,1988.

[4] J.M.C. Clark and R.J. Cameron, The maximum rate of convergence of discrete approximations for stochastic differential equations, *Lecture Notes in Control and Infor Sci,* **25**,Springer-V.,1980,162-171.

[5] P.E. Kloeden and E. Platen, *Numerical Solution of Stochstic Differential Equations,* Springer-Verlag, Berlin,1992.

[6] Y. Saito and T. Mitsui, to appear in *Ann. Inst. Statis. Math.* (1993).

[7] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, *Tech. Res. Rep. No.9202, Dept. of Info. Eng., Nagoya Univ.,* (1992).

[8] Y. Saito and T. Mitsui, to appear in *World Scientific Series on Applicable Analyssis,* **2,** (1993).